TIME PAYMENTS AND SUPERANNUATION

This tutorial was developed by Patrick Curteis.

These questions are not all that difficult. Just set each question out the same way, and work through to the final answer.

Simple Time Payments Question

A man borrows $7000 to buy a new car. He has to pay off the loan over 4 years in equal monthly payments. If the interest rate is 15% p.a. and is compounded monthly, find the amount of each payment.

Answer

Set the solution out in the normal way...

Loan to be repaid in 4 years or 48 months.
Interest rate is 15% p.a. or 1.25% per month.
Let M be the monthly payment.
Let A_n be the amount owing on the loan **after** n monthly payments.

Amount owing after 1 month, before repayment
$= 7000(1.0125)^1$
Amount owing after 1 month, **after** repayment
$A_1 = 7000(1.0125)^1 - M$

Amount owing after 2 months, before repayment
$= [7000(1.0125)^1 - M](1.0125)^1$
$= 7000(1.0125)^2 - M(1.0125)^1$
Amount owing after 2 months, **after** repayment
$A_2 = 7000(1.0125)^2 - M(1.0125)^1 - M$

Amount owing after 3 months, before repayment
$= [7000(1.0125)^2 - M(1.0125)^1 - M](1.0125)^1$
$= 7000(1.0125)^3 - M(1.0125)^2 - M(1.0125)^1$
Amount owing after 3 months, **after** repayment
$A_3 = 7000(1.0125)^3 - M(1.0125)^2 - M(1.0125)^1 - M$

Therefore:
$A_{48} = 7000(1.0125)^{48} - M(1.0125)^{47} - M(1.0125)^{46} - \ldots - M(1.0125)^1 - M$
$= 7000(1.0125)^{48} - M[(1.0125)^{47} + (1.0125)^{46} + \ldots + (1.0125)^1 + 1]$
$= 7000(1.0125)^{48} - M[1 + (1.0125)^1 + \ldots + (1.0125)^{46} + (1.0125)^{47}]$
This is a sum of a GP
a = 1, r = 1.0125, n = 48
$$= 7000(1.0125)^{48} - M\left[\frac{a(r^n - 1)}{r - 1}\right]$$

$$= 7000(1.0125)^{48} - M\left[\frac{1(1.0125^{48} - 1)}{1.0125 - 1}\right]$$

Since $A_{48} = 0$:

$$M\left[\frac{1(1.0125^{48} - 1)}{1.0125 - 1}\right] = 7000(1.0125)^{48}$$

$$65.22838824 \times M = 12707.48397$$

$$M = 194.8152378$$

$$\approx \$194.82$$

Slightly Harder Time Payments Question

Anne borrows \$20000 from City Credit at 12% p.a. interest. She pays it back at regular monthly intervals over 4 years. However, because she is a good customer she is given two months interest free.

(i) If the amount of each monthly payment is M, find, in terms of M:
  (a) the amount she owes after the second payment.
  (b) the amount she owes after the fourth payment.

(ii) How much does she owe after 48 payments?

(iii) Find the amount of each monthly payment.

Answer

Don't worry about answering (i), (ii) or (iii) for the moment. Set the solution out in the same way, month by month, looking for a pattern to appear.

Loan to be repaid in 4 years or 48 months.
Interest rate is 12% p.a. or 1% per month.
Let M be the monthly payment.
Let A_n be the amount owing on the loan **after** n monthly payments.

Amount owing after 1 month (interest free), before repayment
$= 20000$
Amount owing after 1 month (interest free), **after** repayment
$A_1 = 20000 - M$

Amount owing after 2 months (interest free), before repayment
$= 20000 - M$
Amount owing after 2 months (interest free), **after** repayment
$A_2 = 20000 - M - M$
$= 20000 - 2M$

Amount owing after 3 months, before repayment
$= [20000 - 2M](1.01)^1$
$= 20000(1.01)^1 - 2M(1.01)^1$
Amount owing after 3 months, **after** repayment

$A_3 = 20000(1.01)^1 - 2M(1.01)^1 - M$

Amount owing after 4 months, before repayment

$= [20000(1.01)^1 - 2M(1.01)^1 - M](1.01)^1$

$= 20000(1.01)^2 - 2M(1.01)^2 - M(1.01)^1$

Amount owing after 4 months, **after** repayment

$A_4 = 20000(1.01)^2 - 2M(1.01)^2 - M(1.01)^1 - M$

Amount owing after 5 months, before repayment

$= [20000(1.01)^2 - 2M(1.01)^2 - M(1.01)^1 - M](1.01)^1$

$= 20000(1.01)^3 - 2M(1.01)^3 - M(1.01)^2 - M(1.01)^1$

Amount owing after 5 months, **after** repayment

$A_5 = 20000(1.01)^3 - 2M(1.01)^3 - M(1.01)^2 - M(1.01)^1 - M$

Therefore:

$A_{48} = 20000(1.01)^{46} - 2M(1.01)^{46} - M(1.01)^{45} - M(1.01)^{44} - \ldots - M(1.01)^1 - M$

$= 20000(1.01)^{46} - 2M(1.01)^{46} - M[(1.01)^{45} + (1.01)^{44} + \ldots + (1.01)^1 + 1]$

$= 20000(1.01)^{46} - 2M(1.01)^{46} - M[1 + (1.01)^1 + \ldots + (1.01)^{44} + (1.01)^{45}]$

This is a sum of a GP

a = 1, r = 1.01, n = 46

$$= 20000(1.01)^{46} - 2M(1.01)^{46} - M\left[\frac{a(r^n - 1)}{r - 1}\right]$$

$$= 20000(1.01)^{46} - 2M(1.01)^{46} - M\left[\frac{1(1.01^{46} - 1)}{1.01 - 1}\right]$$

Since $A_{48} = 0$:

$$2M(1.01)^{46} + M\left[\frac{1(1.01^{46} - 1)}{1.01 - 1}\right] = 20000(1.01)^{46}$$

$$M\left[2(1.01)^{46} + \frac{1(1.01^{46} - 1)}{1.01 - 1}\right] = 20000(1.01)^{46}$$

$$61.20680318 \times M = 31609.17709$$

$$M = 516.4324136$$

$$\approx \$516.43$$

This looks like a lot of work, especially when we haven't even answered (i), (ii) or (iii)! But look carefully!

The answers to part (i) are already in our working. A_2 is the amount owing after the second payment and A_4 is the amount owing after the fourth payment. The answer to part (ii) is simply 0. The answer to part (iii) is found at the end of our working.

So underneath our working we can summarise our answers:

(i) (a) Amount owing after second payment = $[20000 – 2M]

(b) Amount owing after fourth payment = $\$[20000(1.01)^2 - 2M(1.01)^2 - M(1.01)^1 - M]$

(ii) Amount owing after 48 payments = 0

(iii) M = $516.43

Challenging Time Payments Question (HSC 1990)

A farmer borrows $80 000 to purchase new machinery. The interest is calculated monthly at the rate of 2% per month, and is compounded each month. The farmer intends to repay the loan with interest in two equal annual instalments of $M at the end of the first and second years.

(i) How much does the farmer owe at the end of the first month?

(ii) Write an expression involving M for the total amount owed by the farmer after 12 months, just after the first instalment of $M has been paid.

(iii) Find an expression for the amount owed at the end of the second year and deduce that $M = \dfrac{80000 \times (1.02)^{24}}{(1.02)^{12} + 1}$.

(iv) What is the total interest over the two year period?

Answer

Again, don't worry about answering (i), (ii), (iii) or (iv) for the moment. Set the solution out in the same way, month by month, looking for a pattern to appear.

Loan to be repaid in 2 years or 24 months.
Interest rate is 2% per month.
Let M be the annual payment.
Let A_n be the amount owing on the loan **after** n annual payments.

Amount owing after 1 month, before repayment
$= 80000(1.02)^1$
Note that there is no monthly payment, only two annual payments.
Amount owing after 2 months, before repayment
$= 80000(1.02)^2$
Amount owing after 3 months, before repayment
$= 80000(1.02)^3$
And so, amount owing after 12 months, before repayment
$= 80000(1.02)^{12}$
Amount owing after 12 months, **after** repayment
$A_1 = 80000(1.02)^{12} - M$

Amount owing after 13 months, before repayment
$= [80000(1.02)^{12} - M](1.02)^1$
$= 80000(1.02)^{13} - M(1.02)^1$
Amount owing after 14 months, before repayment
$= [80000(1.02)^{13} - M(1.02)^1](1.02)^1$
$= 80000(1.02)^{14} - M(1.02)^2$
Amount owing after 15 months, before repayment

$= [80000(1.02)^{14} – M(1.02)^2](1.02)^1$
$= 80000(1.02)^{15} – M(1.02)^3$

And so, amount owing after 24 months, before repayment
$= 80000(1.02)^{24} – M(1.02)^{12}$

Amount owing after 24 months, **after** repayment
$A_2 = 80000(1.02)^{24} – M(1.02)^{12} – M$

Since $A_2 = 0$:

$$M(1.02)^{12} - M = 80000(1.02)^{24}$$

$$M\left[(1.02)^{12} + 1\right] = 80000(1.02)^{24}$$

$$M = \frac{80000(1.02)^{24}}{(1.02)^{12} + 1}$$

$$M = 56728.95203$$

$$\approx \$56728.95$$

Once again, the hard work will pay off in the end when we answer each part.

(i) Amount owing at end of first month $= 80000(1.02)^1$
$= \$81600$

(ii) Amount owing after 12 months $= \$[80000(1.02)^{12} – M]$

(iii) Amount owing at end of second year = $0
Expression for M is shown in working above.

(iv) Amount borrowed = $80000
Amount repaid $= 2 \times 56728.95$
= $113457.90
Total interest = 113457.90 – 80000
= $33457.90

Notice that there were no GP's used in this question. Instead a sound understanding of the process of time payments was required!

Look out for "trick questions"!

Some questions appear to be much harder than they really are. If you read through the question very carefully, you will see that the question can be answered simply be setting the solution out as if it was a time payments question.

Example 1 (HSC 1993)
Ezzat invests $50 000 in an account which earns 8% interest, compounded annually. He intends to withdraw $M at the end of each year, immediately after the interest has been paid. He wishes to be able to do this for exactly 20 years, so that the account will then be empty.

This can be set up as a time payments solution where a loan of $50 000 is repaid over 20 years in equal yearly repayments of $M, with an interest charge of 8% p.a., compounded annually.

Example 2

The population of a certain town at the beginning of 2000 was 100 000. The population increases (due to births and new arrivals) by 12% each year, but also decreases (due to deaths and departures) by E people each year.

This can be set up as a time payments solution where the starting value is 100 000. The "interest charge" is 12% p.a. and the "annual payment" is E. The interesting thing about this question is that the population may not diminish to zero at all. The "balance" might always be increasing!

Simple Superannuation Question (HSC 1984)

On January 1, 1957, a person joins a superannuation fund by investing $3000 at 9% p.a. compound interest. A similar amount is invested at the beginning of each subsequent year until the person retires on December 31, 1984.

(i) Show that the accumulated value of the investment at the date of retirement is $369 406, correct to the nearest dollar.

(ii) If this amount is taken as a lump sum payment, which is taxable at the rate of 30 cents in every dollar in excess of $50 000, then how much does the person receive after tax?

Answer

Set the solution out in the normal way...

Let A_n be the amount which the nth investment of $3000 accumulates.

First $3000 is invested at 9% p.a. for 28 years.
$\therefore\ A_1 = 3000(1.09)^{28}$

Second $3000 is invested at 9% p.a. for 27 years.
$\therefore\ A_2 = 3000(1.09)^{27}$

Third $3000 is invested at 9% p.a. for 26 years.
$\therefore\ A_3 = 3000(1.09)^{26}$

Final $3000 is invested at 9% p.a. for 1 year.
$\therefore\ A_{28} = 3000(1.09)^1$

Total amount $= A_1 + A_2 + A_3 + \ldots + A_{28}$

$= 3000(1.09)^{28} + 3000(1.09)^{27} + 3000(1.09)^{26} + \ldots + 3000(1.09)^1$

$= 3000[(1.09)^{28} + (1.09)^{27} + (1.09)^{26} + \ldots + (1.09)^1]$

$= 3000[(1.09)^1 + (1.09)^2 + \ldots + (1.09)^{27} + (1.09)^{28}]$

This is a sum of a GP
$a = 1.09,\ r = 1.09,\ n = 28$

$$= 3000\left[\frac{a(r^n - 1)}{r - 1}\right]$$

$$= 3000\left[\frac{1.09(1.09^{28} - 1)}{1.09 - 1}\right]$$

$= 3000 \times 123.1353565$
$= 369406.0694$
$\approx \$369406$

This gives the answer to part (i). For part (ii), remember how you did tax tables in Year 9 and Year 10?

Tax payable $= 0.30 \times (369406 - 50000)$
$= \$95821.80$
$\therefore$ Amount received after tax $= 369406 - 95821.80$
$= \$273584.20$

Slightly Harder Superannuation Question (HSC 1987)

A person invests \$800 at the beginning of each year in a superannuation fund. Compound interest is paid at 10% per annum on the investment. The first \$800 is to be invested at the beginning of 1988 and the last is to be invested at the beginning of 2017. Calculate to the nearest dollar:

(i) the amount to which the 1988 investment will have grown by the beginning of 2018.

(ii) the amount to which the total investment will have grown by the beginning of 2018.

Answer

After you have read through the question and understood it, let's not worry about trying to do part (i) or (ii) and just set the solution out as above.

Let A_n be the amount which the nth investment of \$800 accumulates.

First \$800 is invested at 10% p.a. for 30 years.
$\therefore A_1 = 800(1.10)^{30}$

Second \$800 is invested at 10% p.a. for 29 years.
$\therefore A_2 = 800(1.10)^{29}$

Third \$800 is invested at 10% p.a. for 28 years.
$\therefore A_3 = 800(1.10)^{28}$

Final \$800 is invested at 10% p.a. for 1 year.
$\therefore A_{30} = 800(1.10)^1$

Total amount $= A_1 + A_2 + A_3 + ... + A_{30}$
$= 800(1.10)^{30} + 800(1.10)^{29} + 800(1.10)^{28} + ... + 800(1.10)^1$
$= 800[(1.10)^{30} + (1.10)^{29} + (1.10)^{28} + ... + (1.10)^1]$
$= 800[(1.10)^1 + (1.10)^2 + ... + (1.10)^{29} + (1.10)^{30}]$

This is a sum of a GP
$a = 1.10$, $r = 1.10$, $n = 30$

$$= 800\left[\frac{a(r^n - 1)}{r - 1}\right]$$

$$= 800\left[\frac{1.10(1.10^{30}-1)}{1.10-1}\right]$$
$= 800 \times 180.943425$
$= \$144754.74$
$\approx \$144755$

Notice that this is identical to the previous example, except some of the numbers have changed.

Now let's get back to worrying about parts (i) and (ii). You can see that we have already answered part (ii), so just put (ii) at the start of your working, and maybe draw a box around the final answer for clarity.

What about part (i). Well, the answer to (i) is actually A_1 above. So underneath this mass of working we could write the following:

(i) Amount which 1988 investment grows to $= A_1$
$= 800(1.10)^{30}$
$= 13959.52182$
$\approx \$13960$

Don't be concerned that your answers are out of order! Make sure you label the answers clearly!
