MATHEMATICAL INDUCTION

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INTRODUCTION: STORY TIME

A budgie flew in the window of the classroom and landed on Grant's shoulder, turned to the class and said: *If I'm blue, then the next budgie that flies in the window is blue.*
Another budgie flew in the window of the classroom and landed on Rachel's shoulder, turned to the class and said: *If I'm blue, then the next budgie that flies in the window is blue.*
Another budgie flew in the window of the classroom and landed on Matthew's shoulder, turned to the class and said: *If I'm blue, then the next budgie that flies in the window is blue.*
And this continued.
(All these budgies have been trained to talk and they all tell the truth.)

Discussion: Are all the budgies blue ?

Demonstration: The domino effect

Examples

1. *Sum of a series* (HSC 1991)

   Use mathematical induction to prove that, for all positive integers n,

$$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1.$$

2. *Divisibility* (HSC 1985)

   Use the Principle of Mathematical Induction to prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

3. *Inequality* (Franklin, Preece & Grunseit)

   Prove by the Principle of Mathematical Induction that $2^n > n^2$ for all $n > 4$.

4. *Formula for* ${}^nC_r$ (Standard bookwork for the 3 Unit course)

   Prove by the Principle of Mathematical Induction that:

$${}^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\times\ldots\times r}$$

   for $1 < r < n$ (proof of binomial coefficients). It can be assumed that you have already proved that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$ for $1 < r < (n-1)$.

5. *Factorisation formula*

   Prove by the Principle of Mathematical Induction that:

$$x^n - c^n$$
$$= (x - c)(x^{n-1} + x^{n-2}c + \ldots + xc^{n-2} + c^{n-1})$$

   for all positive integers n.

6. *Harder 3 Unit topics in the 4 Unit course*

   (4 Unit Syllabus)

   (a) Prove that the angle sum of an n–sided figure is equal to $(2n - 4)$ right angles.

   (b) A sequence $\{u_n\}$ is such that

   $u_{n+3} = 6u_{n+2} - 5u_{n+1}$, and $u_1 = 2$, $u_2 = 6$.

   Prove that $u_n = 5^{n-1} + 1$.

(c) Question 8 in 4 Unit HSC 1985

(i) Show that for $k \geq 0$,

$2k + 3 > 2\sqrt{\{(k+1)(k+2)\}}.$

(ii) Hence prove that for $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2\left[\sqrt{(n+1)} - 1\right].$$

(iii) Is it a true statement that, for all positive integers N, $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$?

Give reasons for your answer.

(d) Question 8 in 4 Unit HSC 1981

Using induction, show that for each positive integer n there are unique positive integers p_n and q_n, such that

$$\left(1+\sqrt{2}\right)^n = p_n + q_n.$$

Show also that $p_n^2 - 2q_n^2 = (-1)^n$.

7. *4 Unit applications*

Prove De Moivre's theorem

$$(\cos q + i \sin q)^n = \cos nq + i \sin nq$$

for positive integers n, and for negative integers n.

8. *Just for fun* (3 Unit HSC 1972, Question 9)

(a) Write down an expression for $\cos(a + b)$ and hence prove that $\cos(2q) = 1 - 2\sin^2 q$.

(b) Prove the identity

$$\frac{\cos y - \cos(y + 2q)}{2 \sin q} = \sin(y + q).$$

(c) Use mathematical induction and the result of part (b) to prove the identity:

$\sin q + \sin 3q + \sin 5q + \ldots + \sin(2n - 1)q$

$$= \frac{1 - \cos 2nq}{2 \sin q}.$$

9. *More fun* (3 Unit HSC 1984, Question 7)

It is given that $A > 0$, $B > 0$ and n is a positive integer.

(a) Divide $A^{n+1} - A^nB + B^{n+1} - B^nA$

by $A - B$, and deduce that

$A^{n+1} + B^{n+1} > A^nB + B^nA.$

(b) Using (a), show by mathematical induction that

$$\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}.$$

Solutions

Example 1

$1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$

for all positive integers n.

(α) Prove true for $n = 1$.

LHS = 1

RHS = $2^1 - 1 = 1$ = LHS.

True for $n = 1$.

(β) Assume true for $n = k$, that is,

$$1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1,$$

then prove true for $n = k + 1$, that is, prove

$$1 + 2 + 4 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1.$$

Proof: LHS $= 1 + 2 + 4 + \ldots + 2^{k-1} + 2^k$

$= 2^k - 1 + 2^k$

$= 2.2^k - 1$

$= 2^{k+1} - 1$

= RHS.

(χ) If the statement is true for n = k, then it is also true for $n = k + 1$. Since the statement is true for $n = 1$, it follows that it must also be true for $n = 2$; and since it is true for $n = 2$, it must also be true for $n = 3$; and so on. Therefore the statement is true for all positive integers n.

Example 2

Prove that $5^n + 2(11^n)$ is a multiple of 3 for all positive integers n.

(α) Prove true for n =1.

$5^1 + 2(11^1) = 27$ which is a multiple of 3,

$\therefore$ true for $n = 1$.

(β) Assume true for $n = k$, that is,

$5^k + 2\,(11^k) = 3Q$ where $Q\ \varepsilon\ J$,

then prove true for $n = k + 1$, that is, prove

$5^{k+1} + 2\,(11^{k+1}) = 3Q'$ where $Q'\ \varepsilon\ J$.

Proof: $5^{k+1} + 2.11^{k+1}$

$= 5.5^k + 2.11.11^k$

$= 5.5^k + 5.2.11^k - 5.2.11^k = 2.11.11^k$

$= 5\,(5^k + 2.11^k) - 10.11^k + 22.11^k$

$= 5.3Q + 12.11^k$

$= 3\,(5Q + 4.11^k)$

$= 3Q'$, where $Q' = 5Q + 4.11^k$.

Now $Q' \varepsilon J$, since 5, Q, 4, 11 and $K \varepsilon J$ and the set of integers J is closed under addition and multiplication.

(χ) Same conclusion as Example 1.

Example 3

Prove $2^n > n^2$ for all integral $n > 4$.

(α) Prove true for $n = 5$.

LHS = 2 = 32

RHS = 5^2 = 25

LHS > RHS.

∴ true for $n = 5$.

(β) Assume true for $n = k$, that is, $2^k > k^2$, then prove true for $n = k + 1$, that is, prove $2^{k+1} > (k + 1)^2$.

Proof: $2^k > k^2$

$2.2^k > 2k^2$

$2^{k+1} > k^2 + k^2$.

But $k^2 > 2k + 1$, for $k \geq 3$,

(can be demonstrated graphically)

∴ $2^{k+1} > k^2 + 2k + 1$, for $k \geq 3$,

∴ $2^{k+1} > (k + 1)^2$, for $k \geq 3$.

(χ) If the statement is true for $n = k$, then it is also true for $n = k + 1$.

Since the statement is true for $n = 5$, it follows that it must also be true for $n = 6$; and since it is true for $n = 6$, it must also be true for $n = 7$; and so on.

Therefore the statement is true for all positive integral $n > 4$.

Example 4

Prove $^nC_r = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\times\ldots\times r}$

for $1 < r < n$.

We will assume that

$^nC_0 = 1$, $^nC_n = 1$, $^nC\text{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

have been proved (refer to page 84 of the syllabus).

(α) Prove true for $n = 2$.

$(1 + x)^2 = 1 + 2x + x^2$

$= {}^2C_0 + {}^2C_1x + {}^2C_2x^2$

$^2C_0 = 1$, $^2C_1 = \ = 2$, $^2C_2 = 1$,

∴ the statement is true for $n = 2$.

(β) Assume the statement is true for n = k – 1:

$$^{k-1}C_r = \frac{(k-1)(k-2)\ldots(k-r)}{1\times 2\times 3\times\ldots\times r},$$

then prove true for $n = k$, that is, prove

$$^{k}C_r = \frac{k(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times r}.$$

Proof:

$$^{k-1}C_r = \frac{(k-1)(k-2)\ldots(k-r)}{1\times 2\times 3\times\ldots\times r}$$

$$^{k-1}C_{r-1} = \frac{(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times(r-1)},$$

But $^{k-1}C_{r-1} + {}^{k-1}C_{\mathrm{r}} = {}^{k}C_r$

$$\therefore\quad {}^{k}C_r = \frac{(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times(r-1)} + \frac{(k-1)(k-2)\ldots(k-r+1)(k-r)}{1\times 2\times 3\times\ldots\times(r-1)\times r}$$

$$\therefore\quad {}^{k}C_r = \frac{(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times(r-1)}\left[1+\frac{(k-r)}{r}\right]$$

$$= \frac{(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times(r-1)}\left[\frac{(r+k-r)}{r}\right]$$

$$= \frac{k(k-1)(k-2)\ldots(k-r+1)}{1\times 2\times 3\times\ldots\times(r-1)\times r}$$

= RHS.

(c) If the statement is true for $n = k – 1$, then it is also true for $n = k$. Since it is true for $n = 2$, it must also be true for $n = 3$; and so on. Therefore the statement is true for positive integers n.

Special cases:

$^{1}C_0 = 1$, $^{1}C_1 = 1$, $^{n}C_0 = 1$, $^{n}C_n = 1$

$$\therefore\quad {}^{n}C_r = \frac{n!}{(n-r)!\,r!},\quad \text{for } 1 \le r \le n,$$

and all positive integers n.

Example 5

Prove that, for all positive integers n,

$(x^n – c^n) = (x – c)\,(x^{n–1} + x^{n–2}c + \ldots + xc^{n–2} + c^{n–1})$.

(α) Prove true for $n = 1$.

LHS $= x – c$

RHS $= x – c$, since $x – c$ cannot be factorised.

Prove true for $n = 2$.

LHS $= x^2 – c^2$

RHS $= (x – c)\,(x + c)$

$= x^2 – c^2$ = RHS.

True for $n = 2$.

(β) Assume true for $n = k$,

that is, $(x^k - c^k)$

$= (x - c)(x^{k-1} + x^{k-2}c + ... + xc^{k-2} + c^{k-1})$,

then prove true for $n = k + 1$,

that is, prove $(x^{k+1} - c^{k+1})$

$= (x - c)(x^k + x^{k-1}c + ... + xc^{k-1} + c^k)$.

Proof:

LHS $= x^{k+1} - c^{k+1}$

$= x^{k+1} - x^k c + x^k c - c^{k+1}$

$= x^k(x - c) + c(x^k - c^k)$

$= x^k(x - c) + c(x - c)(x^{k-1} + x^{k-2}c + \equiv$

$+ xc^{k-2} + c^{k-1})$

$= (x - c)(x^k + x^{k-1}c + x^{k-2}c^2 + \equiv$

$+ xc^{k-1} + c^k)$

= RHS.

(χ) *Conclusion*:

Similar to Example 1, but commencing at $n = 2$.

Example 6(a)

Prove that the angle sum of an n–sided figure is equal to $2n - 4$ right angles.

(a) Prove true for $n = 3$.

The angle sum of a triangle = 180° = 2 right angles.

$2n - 4 = 2 \times 3 - 4 = 2$,

$\therefore$ true for $n = 3$.

(b) Assume true for $n = k$,

that is, the angle sum of a k–sided figure

$= (2k - 4)$ right angles.

Then prove true for $n = k + 1$, that is,

prove the angle sum of a $(k + 1)$ – sided figure

$= (2k - 2)$ right angles.

Proof:

To form a $(k + 1)$ – sided figure from a k–sided figure, replace one side by another two:

Angle sum of $(k + 1)$ – sided figure = angle sum of k–sided figure + angle sum of triangle

$= (2k - 4)$ right angles + 180°.

Angle sum of $(k + 1)$ - sided figure

$= (2k - 4 + 2)$ right angles

$= (2k - 2)$ right angles.

(χ) If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 3$, it follows that it must be true for $n = 4$; and so on.

Therefore the angle sum of an n–sided figure is equal to $(2n - 4)$ right angles for all $n \geq 3$.

Example 6(b)

A sequence (u_n) is such that $u_{n+3} = 6u_{n+2} - 5u_{n+1}$ and $u_1 = 2$, $u_2 = 6$. Prove that $u_n = 5^{n-1} + 1$.

(α) Prove true for $n = 1$.

$5^{1-1} + 1 = 5^{\circ} + 1$

$= 1 + 1 = 2.$

$u_1 = 2$, given,

∴ true for $n = 1$.

Prove true for $n = 2$.

$5^{2-1} + 1 = 5 + 1 = 6.$

$u_2 = 6$, given,

∴ true for $n = 2$.

Prove true for $n = 3$.

$u_3 = 5^{3-1} + 1$

$= 5^2 + 1 = 26.$

$u_3 = 6u_2 - 5u_1$

$= 6 \infty 6 - 5 \infty 2$

$= 36 - 10 = 26$

∴ true for $n = 3$.

(b) Assume true for $n = k$, $n = k + 1$, $n = k + 2$,

that is, $u_k = 5^{k-1} + 1$, $u_{k+1} = 5^k + 1$, $u_{k+2} = 5^{k+1} + 1$.

Then prove true for $n = k + 3$,

that is, prove $u_{k+3} = 5^{k+2} + 1$.

Proof:

$u_{k+3} = 6u_{k+2} - 5u_{k+1}.$

$= 6\,(5^{k+1} + 1) - 5(5^k + 1)$

$= 6 \,.\, 5^{k+1} + 6 - 5 \,.\, 5^k - 5$

$= 6 \,.\, 5^{k+1} - 5^{k+1} + 1$

$= 5 \,.\, 5^{k+1} + 1$

$= 5^{k+2} + 1 = \text{RHS}.$

(c) If the statement is true for $n = k$, $n = k + 1$, and $n = k + 2$, it also true for $n = k + 3$. Since the statement is true for $n = 1$, $n = 2$, and $n = 3$, it follows that the statement is true for $n = 4$, and so on. Therefore the statement is true for all positive integers n.

Example 6(c)

(i) $2k + 3 > 2\sqrt{(k+1)(k+2)}$ for $k \geq 0$.

$(2k + 3)^2 - 4(k + 1)(k + 2)$

$= 4k^2 + 12k + 9 - 4k^2 - 12k - 8$

$= 1$,

$\therefore$ $(2k + 3)^2 > 4(k + 1)(k + 2)$. **(1)**

For $k \geq 0$, $2k + 3 > 0$, and $(k + 1)(k + 2) > 0$.

So taking the positive square root of both sides of **(1):**

$(2k + 3) > 2\sqrt{(k+1)(k+2)}$.

(ii) Prove, by induction, for $n \geq 1$:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \equiv + \frac{1}{\sqrt{n}} > 2\left[\sqrt{n+1} - 1\right]$$

(α) Prove true for $n = 1$.

LHS $= 1$

RHS $= 2\left[\sqrt{2} - 1\right] < 2\left[1\cdot 42 - 1\right]$

< 0.84

< 1,

$\therefore$ true for $n = 1$.

(β) Assume true for $n = k$,

that is, $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{k}} > 2\left[\sqrt{k+1} - 1\right]$

then prove true for $n = k + 1$,

that is, prove $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \equiv + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$> 2\left[\sqrt{k+2} - 1\right]$

Proof: LHS $> 2\left[\sqrt{k+1} - 1\right] + \frac{1}{\sqrt{k+1}}$

$= \frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}}$

$= \frac{2k + 3 - 2\sqrt{k+1}}{\sqrt{k+1}}$.

But $2k + 3 > 2\sqrt{(k+1)(k+2)}$ [part (i)]

$\therefore$ $\frac{2k + 3 - 2\sqrt{k+1}}{\sqrt{k+1}}$

$> \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{k+1}}{\sqrt{k+1}}$

$\therefore$ $> 2\sqrt{k+2} - 2$

$\therefore \quad > 2\left[\sqrt{k+2}-1\right]$

$\therefore \quad \sum_{r=1}^{k+1} \frac{1}{\sqrt{r}} > 2\left[\sqrt{k+2}-1\right]$

(χ) *Conclusion*: The same as Example 1.

iii) Is the statement $\sum_{k=1}^{N} \frac{1}{\sqrt{k}} < 10^{10}$ true for all positive integers N ?

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} > 2\left[\sqrt{N+1}-1\right].$$

Is the number N such that

$$10^{10} = 2\left[\sqrt{N+1}-1\right]?$$

$$\sqrt{N+1} = \frac{10^{10}}{2} + 1,$$

$$N = \left(\frac{10^{10}}{2} + 1\right)^2 - 1,$$

$\therefore$ when $N = \left(\frac{10^{10}}{2} + 1\right)^2 - 1$,

$$\sum_{k=1}^{N} \frac{1}{\sqrt{k}} > 10^{10}.$$

So the original statement is not true.

Example 6(d)

$p_n, q_n \; \varepsilon \; J$, prove that $\left(1+\sqrt{2}\right)^n = p_n + q_n$.

(α) Prove true for $n = 1$.

$\left(1+\sqrt{2}\right)^1 = 1 + \sqrt{2}$

which is in the form $p_1 + q_1\sqrt{2}$,

where $p_1 = 1$, $q_1 = 1$,

$\therefore$ true for $n = 1$.

(β) Assume true for $n = k$.

that is, $\left(1+\sqrt{2}\right)^k = p_k + q_k\sqrt{2}$ where $p_k, q_k \; \varepsilon$ J,

then prove true for $n = k + 1$,

that is, prove $\left(1+\sqrt{2}\right)^{k+1} = p_{k+1} + q_{k+1}\sqrt{2}$,

where $p_{k+1}, q_{k+1} \; \varepsilon \; J$.

Proof: LHS $= \left(1+\sqrt{2}\right)^{k+1}$

$$= \left(1+\sqrt{2}\right)^k \left(1+\sqrt{2}\right)$$
$$= \left(p_k + q_k\sqrt{2}\right)\left(1+\sqrt{2}\right)$$
$$= p_k + q_k\sqrt{2} + q_k\sqrt{2} + 2q_k$$
$$= \left(p_k + 2q_k\right) + \left(p_k + q_k\right)\sqrt{2}$$
$$= p_{k+1} + q_{k+1}\sqrt{2}\ ,$$

where $p_{k+1} = p_k + 2q_k$

and $q_{k+1} = p_k + q_k$,

and p_{k+1} and q_{k+1} ε J, since J is closed under addition and multiplication.

(χ) *Conclusion*: As in Example 1.

Example 7

Prove: De Moivre's theorem for $n \geq 1$:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(α) Prove true for $n = 1$.

$$\text{LHS} = \cos\theta + i\sin\theta$$
$$\text{RHS} = \cos\theta + i\sin\theta = \text{LHS}\ ,$$

∴ true for $n = 1$.

(β) Assume true for $n = k$,

that is , $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

then prove true for $n = k + 1$, that is, $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$.

Proof :
$$\begin{aligned}
\text{LHS} &= (\cos\theta + i\sin\theta)^{k+1} \\
&= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) \\
&= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\
&= \cos k\theta\cos\theta + i\sin\theta\cos k\theta \\
&\quad + i\cos\theta\sin k\theta\sin k\theta\sin\theta \\
&= \cos k\theta\cos\theta\sin k\theta\sin\theta \\
&\quad + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta) \\
&= \cos(k\theta + \theta) + i\sin(k\theta + \theta) \\
&= \cos(k+1)\theta + i\sin(k+1)\theta \\
&= \text{RHS}\ .
\end{aligned}$$

(χ) *Conclusion*: As in Example 1.

Prove: De Moivre's theorem for $n \leq -1$.

(α) Prove true for $n = -1$.

$$\text{LHS} = (\cos q + i \sin q)^{-1}$$
$$= \frac{1}{\cos q + i \sin q} \times \frac{\cos q - i \sin q}{\cos q - i \sin q}$$
$$= \frac{\cos q - i \sin q}{\cos^2 q + \sin^2 q}$$
$$= \cos(-q) + i \sin(-q)$$
$$= \text{RHS}.$$

(β) Assume true for $n = -k$,

that is, $(\cos q + i \sin q)^{-k} = \cos(-kq) + i \sin(-kq)$,

then prove true for $n = -k - 1 = -(k + 1)$,

that is, prove $(\cos q + i \sin q)^{-(k+1)}$

$$= \cos[-(k+1)q] + i \sin[-(k+1)q].$$

Proof:

$$\text{LHS} = (\cos q + i \sin q)^{-k-1}$$
$$= (\cos q + i \sin q)^{-k}(\cos q + i \sin q)^{-1}$$
$$= [\cos(-kq) + i \sin(-kq)][\cos(-q) + i \sin(-q)]$$

(from assumption) (from part (a))

$$= \cos(-kq)\cos(-q) + i \sin(-kq)\cos(-q)$$
$$+ i \cos(-kq)\sin(-q) - \sin(-kq)\sin(-q)$$
$$= \cos(-kq - q) + i \sin(-kq - q)$$
$$= \cos[-(k+1)q] + i \sin[-(k+1)q]$$
$$= \text{RHS}.$$

(χ) If the theorem is true for $n = -k$ it is also true for $n = -k - 1$. Since the theorem is true for $n = -1$, it follows that it must be true for $n = -2$, and so on. So De Moivre's theorem is true for all integers $n \leq -1$.

Example 8

(a) $\cos(a + b) = \cos a \cos b - \sin a \sin b$

$$\cos(2q) = \cos^2 q - \sin^2 q$$
$$= 1 - \sin^2 q - \sin^2 q$$
$$= 1 - 2\sin^2 q.$$

(b) $\dfrac{\cos y - \cos(y + 2q)}{2 \sin q} = \sin(y + q)$.

LHS

$$= \frac{\cos y - \cos(y + 2q)}{2 \sin q}$$
$$= \frac{\cos y - (\cos y \cos 2q - \sin y \sin 2q)}{2 \sin q}$$
$$= \frac{\cos y - \cos y(1 - 2\sin^2 q) + \sin y . 2 \sin q \cos q}{2 \sin q}$$
$$= \frac{\cos y - \cos y + 2\sin^2 q \cos y + 2 \sin q \sin y \cos q}{2 \sin q}$$

= sin q cos y + sin y cos q

= sin (y + q)

= RHS

(c) sin q + sin 3q + sin 5q + ... + sin (2n – 1)q

$$= \frac{1-\cos 2nq}{2\sin q} .$$

(α) Prove true for n = 1.

LHS = sin q.

$$\text{RHS} = \frac{1-\cos 2q}{2\sin q}$$

$$= \frac{1-(1-2\sin^2 q)}{2\sin q}$$

$$= \frac{1-1+2\sin^2 q}{2\sin q} ,$$

= sin q = LHS.

∴ true for n = 1.

(β) Assume true for $n = k$, that is,

sin$q$ + sin 3q + sin 5q + ... + sin (2k – 1)q

$$= \frac{1-\cos 2kq}{2\sin q} .$$

then prove true for $n = k + 1$,

that is, prove:

sin q + sin 3q +... + sin(2k – 1)q + sin(2k + 1)q

$$= \frac{1-\cos 2(k+1)q}{2\sin q} .$$

Proof: LHS $= \frac{1-\cos 2kq}{2\sin q}$ + sin(2k + 1)q

$$= \frac{1-\cos 2kq}{2\sin q} + \sin(2kq + q)$$

$$= \frac{1-\cos 2kq}{2\sin q} + \frac{\cos 2kq - \cos(2kq+2q)}{2\sin q}$$

from part (ii)

$$= \frac{1-\cos 2(k+1)q}{2\sin q}$$

= RHS.

(χ) *Conclusion*: As in Example 1.

Example 9

(a) $A > 0$, $B > 0$, $n \geq 1$

$$\frac{A^{n+1} - A^nB + B^{n+1} - B^nA}{A-B}$$

$$= \frac{A^n(A-B)+B^n(B-A)}{A-B}$$

$= A^n - B^n$.

If $A > B$, then $A^n - B^n > 0$,

$\therefore \frac{A^{n+1} - A^n b + B^{n+1} - B^n A}{A - B} > 0$,

$\therefore A^{n+1} - A^n B + B^{n+1} - B^n A > 0$, since $A - B > 0$,

$\therefore A^{n+1} + B^{n+1} > A^n B + B^n A$.

If $B > A$ then $A^n - B^n < 0$ and $A - B < 0$,

$\therefore \frac{A^{n+1} - A^n B + B^{n+1} - B^n A}{A - B} < 0$,

$\therefore A^{n+1} - A^n B + B^{n+1} - B^n A > 0$, since $A - B < 0$,

$\therefore A^{n+1} + B^{n+1} > A^n B + B^n A$.

(b) Prove by mathematical induction that

$\left(\frac{A+B}{2}\right)^n \le \frac{A^n + B^n}{2}$.

(α) Prove true for $n = 1$.

LHS = $\frac{A+B}{2}$ = RHS,

$\therefore$ true for $n = 1$.

Prove true for $n = 2$.

$$\text{LHS} = \left(\frac{A+B}{2}\right)^2 = \frac{A^2 + 2AB + B^2}{4}$$

$$\frac{A^2 + B^2}{4} + \frac{AB + BA}{4}$$

$$\frac{A^2 + B^2}{4} + \frac{A^2 + B^2}{4} \quad \text{from (a),}$$

$$\therefore \left(\frac{A+B}{2}\right)^2 < \frac{2(A^2 + B^2)}{4}$$

$$\therefore \left(\frac{A+B}{2}\right)^2 < \frac{A^2 + B^2}{2},$$

$\therefore$ true for $n = 2$.

(β) Assume true for $n = k$,

that is, $\left(\frac{A+B}{2}\right)^k < \frac{A^k + B^k}{2}$, ,

then prove true for $n = k + 1$,

that is, $$\left(\frac{A+B}{2}\right)^{k+1} < \frac{A^{k+1}+B^{k+1}}{2} ..$$

Proof $$\left(\frac{A+B}{2}\right)^{k} < \frac{A^{k}+B^{k}}{2}$$

$$\left(\frac{A+B}{2}\right)^{k}\left(\frac{A+B}{2}\right) < \left(\frac{A^{k}+B^{k}}{2}\right)\left(\frac{A+B}{2}\right)$$

$$\left(\frac{A+B}{2}\right)^{k+1} < \frac{A^{k+1}+A^{k}B+B^{k}A+B^{k+1}}{4}$$

But $A^{k}B+B^{k}A < A^{k+1}+B^{k+1}$, from (a),

$$\therefore \left(\frac{A+B}{2}\right)^{k+1} < \frac{A^{k+1}+A^{k+1}+B^{k+1}+B^{k+1}}{4}$$

$$\therefore \left(\frac{A+B}{2}\right)^{k+1} < \frac{A^{k+1}+B^{k+1}}{2}$$

(χ) *Conclusion:*

As in Example 1, but starting $n = 2$.

Special case $n = 1$.